Theorem 1. $\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$.

Proof. Let $f(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$. First, we'll neglect the limits and expand $f(x)$ using the binomial theorem.

$$f_n(x) = \sum_{k=0}^{n}\binom{x}{n}^k \frac{n!}{k!(n-k)!}$$

Now, we will increase n to a sufficiently large value such that $|f_n(x) - \exp(x)| < \epsilon, \forall x$. Once we find such a value, our proof will be complete.

$$|\exp(x) - f_n(x)| = \left|\sum_{k=0}^{n}\left(\frac{x^k}{k!} - \binom{x}{n}^k \frac{n!}{k!(n-k)!}\right) + \sum_{k=n+1}^{\infty}\frac{x^k}{k!}\right|.$$

We will make a slight change to the above equation and add a variable m.

$$|\exp(x) - f_n(x)| = \left|\sum_{k=0}^{m}\left(\frac{x^k}{k!} - \binom{x}{n}^k \frac{n!}{k!(n-k)!}\right) + \sum_{k=m+1}^{n}\left(\frac{x^k}{k!} - \binom{x}{n}^k \frac{n!}{k!(n-k)!}\right) + \sum_{k=n+1}^{\infty}\frac{x^k}{k!}\right|$$

$$\leq \left|\sum_{k=0}^{m}\left(\frac{x^k}{k!} - \binom{x}{n}^k \frac{n!}{k!(n-k)!}\right)\right| + \left|\sum_{k=m+1}^{n}\left(\frac{x^k}{k!} - \binom{x}{n}^k \frac{n!}{k!(n-k)!}\right)\right| + \left|\sum_{k=n+1}^{\infty}\frac{x^k}{k!}\right|$$

$$\leq \left|\sum_{k=0}^{m}\left(\frac{x^k}{k!} - \binom{x}{n}^k \frac{n!}{k!(n-k)!}\right)\right| + \left|\sum_{k=m+1}^{n}\frac{x^k}{k!}\right| + \left|\sum_{k=n+1}^{\infty}\frac{x^k}{k!}\right|$$

$$\leq \left|\sum_{k=0}^{m}\left(\frac{x^k}{k!} - \binom{x}{n}^k \frac{n!}{k!(n-k)!}\right)\right| + \left|\sum_{k=m+1}^{\infty}\frac{x^k}{k!}\right|$$

We will look at the 2 terms individually, starting with the right. Since $\frac{x^k}{k!}$ is eventually decreasing to 0, we can choose m large enough so $\frac{x^m}{m!} < \frac{\epsilon}{4}$ and select an n larger than that.

$$\left|\sum_{k=m+1}^{\infty}\frac{x^k}{k!}\right| \leq \frac{\epsilon}{4}\sum_{k=m+1}^{\infty}2^{-k} < \frac{\epsilon}{2}$$

Now, let's look at the left sum. The following expression will help us understand why the left sum is small.

$$\binom{x}{n}^k \frac{n!}{k!(n-k)!} = \frac{x^k(n(n-1)\dots(n-k+1)}{n^k k!} = \frac{x^k}{k!}\frac{(n(n-1)\dots(n-k+1)}{n^k} \approx \frac{x^k}{k!}$$

This approximation is valid as the terms almost cancel out. Because of this each of the terms will be fairly small so hopefully we can make the entire sum small. A clever observation is $n(n-m+1) \leq (n-c)(n-k+1+c), \forall k$ and positive c, this further means that

$$1 > \frac{(n(n-1)\dots(n-k+1))}{n^k} > \left(\frac{n-m+1}{n}\right)^{m/2} > 0$$

This further implies

$$\left|\sum_{k=0}^{m}\left(\frac{x^k}{k!} - \binom{x}{n}^k \frac{n!}{k!(n-k)!}\right)\right| \leq \left|\sum_{k=0}^{m}\frac{x^k}{k!}\left(1-\left(\frac{n-m+1}{n}\right)^{m/2}\right)\right| = \left|\left(1-\left(\frac{n-m+1}{n}\right)^{m/2}\right)\sum_{k=0}^{m}\frac{x^k}{k!}\right|$$

From this, we know that $\exp(x)$ is finite everywhere, $\exists c$ such that c is greater than all the partial sums of $\exp(x)$, therefore

$$\left|\left(1-\left(\frac{n-m+1}{n}\right)^{m/2}\right)\sum_{k=0}^{m}\frac{x^k}{k!}\right| < c\left(1-\left(\frac{n-m+1}{n}\right)^{m/2}\right)$$

We can now use Bernoulli's inequality to say that

$$1 > \left(\frac{n-m+1}{n}\right)^{m/2} = \left(1-\frac{m+1}{n}\right)^{m/2} > 1 - \frac{m^2+m}{2n}$$

Using this,

$$c\left(1-\left(\frac{n-m+1}{n}\right)^{m/2}\right) < \frac{c(m^2+m)}{2n}$$

We fixed m earlier, and we can use this to find out how large n must be, we get

$$n > \frac{c(m^2+m)}{\epsilon}$$

Ergo, putting all of this together

$$|\exp(x) - f_n(x)| \leq \left|\sum_{k=0}^{m}\left(\frac{x^k}{k!} - \binom{x}{n}^k \frac{n!}{k!(n-k)!}\right)\right| + \left|\sum_{k=m+1}^{\infty}\frac{x^k}{k!}\right| < \epsilon$$

This shows that $f_n \to \exp(x)$ pointwise, in other words, $\exp(x) = f(x)$.

QED.